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8-3

Study Guide

Vectors in Three-Dimensional Space

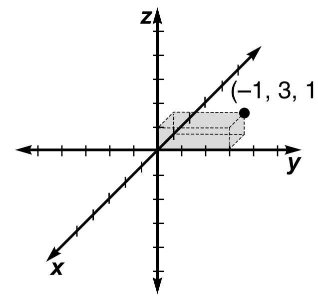
Ordered triples, like ordered pairs, can be used to represent vectors. Operations on vectors respresented by ordered triples are similar to those on vectors represented by ordered pairs. For example, an extension of the formula for the distance between two points in a plane allows us to find the distance between two points in space.

Example 1 **Locate the point at (−1, 3, 1).**

Locate −1 on the x-axis, 3 on the y-axis, and 1 on the z-axis.

Now draw broken lines for parallelograms to represent the three planes.

The planes intersect at (−1, 3, 1).



Example 2 **Write the ordered triple that represents the vector from $X(-4, 5, 6)$ to $Y(-2, 6, 3)$. Then find the magnitude of $\overrightarrow{XY}$.**

$$\begin{aligned}\overrightarrow{XY} &= (-2, 6, 3) - (-4, 5, 6)\\ &= \langle -2 - (-4), 6 - 5, 3 - 6\rangle\\ &= \langle 2, 1, -3\rangle\end{aligned}$$

$$\begin{aligned}|\overrightarrow{XY}| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\\ &= \sqrt{[-2 - (-4)]^2 + (6 - 5)^2 + (3 - 6)^2}\\ &= \sqrt{(2)^2 + (1)^2 + (-3)^2}\\ &= \sqrt{14} \text{ or } 3.7\end{aligned}$$

Example 3 **Find an ordered triple that represents $2\vec{\mathbf{s}} + 3\vec{\mathbf{t}}$ if $\vec{\mathbf{s}} = \langle 5, -1, 2\rangle$ and $\vec{\mathbf{t}} = \langle 4, 3, -2\rangle$.**

$$\begin{aligned}2\vec{\mathbf{s}} + 3\vec{\mathbf{t}} &= 2\langle 5, -1, 2\rangle + 3\langle 4, 3, -2\rangle\\ &= \langle 10, -2, 4\rangle + \langle 12, 9, -6\rangle\\ &= \langle 22, 7, -2\rangle\end{aligned}$$

Example 4 **Write $\overrightarrow{AB}$ as the sum of unit vectors for $A(5, -2, 3)$ and $B(-4, 2, 1)$.**

First express $\overrightarrow{AB}$ as an ordered triple. Then write the sum of the unit vectors $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$.

$$\begin{aligned}\overrightarrow{AB} &= (-4, 2, 1) - (5, -2, 3)\\ &= \langle -4 - 5, 2 - (-2), 1 - 3\rangle\\ &= \langle -9, 4, -2\rangle\\ &= -9\vec{\mathbf{i}} + 4\vec{\mathbf{j}} - 2\vec{\mathbf{k}}\end{aligned}$$

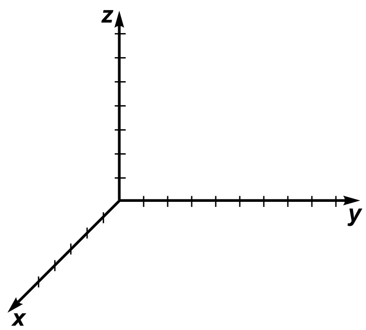
8-3

Practice

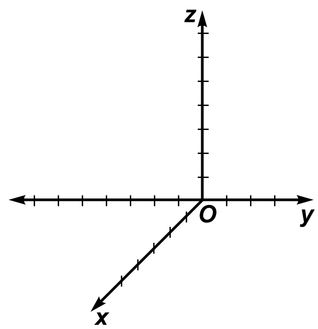
Vectors in Three-Dimensional Space

Locate point B in space. Then find the magnitude of a vector from the origin to B.

1. $B(4, 7, 6)$



2. $B(4, -2, 6)$



Write the ordered triple that represents $\overrightarrow{AB}$. Then find the magnitude of $\overrightarrow{AB}$.

3. $A(2, 1, 3), B(-4, 5, 7)$

4. $A(4, 0, 6), B(7, 1, -3)$

5. $A(-4, 5, 8), B(7, 2, -9)$

6. $A(6, 8, -5), B(7, -3, 12)$

Find an ordered triple to represent $\vec{u}$ in each equation if $\vec{v} = \langle 2, -4, 5\rangle$ and $\vec{w} = \langle 6, -8, 9\rangle$.

7. $\vec{\mathbf{u}} = \vec{\mathbf{v}} + \vec{\mathbf{w}}$

8. $\vec{\mathbf{u}} = \vec{\mathbf{v}} - \vec{\mathbf{w}}$

9. $\vec{\mathbf{u}} = 4\vec{\mathbf{v}} + 3\vec{\mathbf{w}}$

10. $\vec{\mathbf{u}} = 5\vec{\mathbf{v}} - 2\vec{\mathbf{w}}$

11. ***Physics*** Suppose that the force acting on an object can be expressed by the vector $\langle 85, 35, 110\rangle$, where each measure in the ordered triple represents the force in pounds. What is the magnitude of this force?

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Enrichment

Basis Vectors in Three-Dimensional Space

The expression $\vec{\mathbf{v}} = r\vec{\mathbf{u}} + s\vec{\mathbf{w}} + t\vec{\mathbf{z}}$, the sum of three vectors each multiplied by scalars, is called a **linear combination** of the vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{w}}$, and $\vec{\mathbf{z}}$.

Every vector $\vec{\mathbf{v}} \in v_3$ can be written as a linear combination of any three nonparallel vectors. The three nonparallel vectors, which must be linearly independent, are said to form a **basis** for v_3, which contains all vectors having 1 column and 3 rows.

Example **Write the vector $\vec{\mathbf{v}} = \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix}$ as a linear combination of the vectors $\vec{\mathbf{u}} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$, $\vec{\mathbf{w}} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, and $\vec{\mathbf{z}} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.**

$$\begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix} = r\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} r + s - t \\ 3r - 2s - t \\ r + s + t \end{pmatrix}$$

$$\begin{aligned} -1 &= r + s - t \\ -4 &= 3r - 2s - t \\ 3 &= r + s + t \end{aligned}$$

Solving the system of equations yields the solution $r = 0$, $s = 1$, and $t = 2$. So, $\vec{\mathbf{v}} = \vec{\mathbf{w}} + 2\vec{\mathbf{z}}$.

Write each vector as a linear combination of the vectors $\vec{u}$, $\vec{w}$, and $\vec{z}$.

1. $\vec{\mathbf{v}} = \begin{pmatrix} -6 \\ -2 \\ 2 \end{pmatrix}$, $\vec{\mathbf{u}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{\mathbf{w}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\vec{\mathbf{z}} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

2. $\vec{\mathbf{v}} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix}$, $\vec{\mathbf{u}} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\vec{\mathbf{w}} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, and $\vec{\mathbf{z}} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$

3. $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\vec{\mathbf{u}} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\vec{\mathbf{w}} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, and $\vec{\mathbf{z}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

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Study Guide

Perpendicular Vectors

Two vectors are perpendicular if and only if their **inner product** is zero.

Example 1 **Find each inner product if $\vec{u} = \langle 5, 1\rangle$, $\vec{v} = \langle -3, 15\rangle$, and $\vec{w} = \langle 2, -1\rangle$. Is either pair of vectors perpendicular?**

a. $\vec{u} \cdot \vec{v}$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 5(-3) + 1(15) \\ &= -15 + 15 \\ &= 0 \end{aligned}$$

$\vec{u}$ and $\vec{v}$ are perpendicular.

b. $\vec{v} \cdot \vec{w}$

$$\begin{aligned} \vec{v} \cdot \vec{w} &= -3(2) + 15(-1) \\ &= -6 + (-15) \\ &= -21 \end{aligned}$$

$\vec{v}$ and $\vec{w}$ are not perpendicular.

Example 2 **Find the inner product of $\vec{r}$ and $\vec{s}$ if $\vec{r} = \langle 3, -1, 0\rangle$ and $\vec{s} = \langle 2, 6, 4\rangle$. Are $\vec{r}$ and $\vec{s}$ perpendicular?**

$$\begin{aligned} \vec{r} \cdot \vec{s} &= (3)(2) + (-1)(6) + (0)(4) \\ &= 6 + (-6) + 0 \\ &= 0 \end{aligned}$$

$\vec{r}$ and $\vec{s}$ are perpendicular since their inner product is zero.

Unlike the inner product, the **cross product** of two vectors is a vector. This vector does not lie in the plane of the given vectors but is perpendicular to the plane containing the two vectors.

Example 3 **Find the cross product of $\vec{v}$ and $\vec{w}$ if $\vec{v} = \langle 0, 4, 1\rangle$ and $\vec{w} = \langle 0, 1, 3\rangle$. Verify that the resulting vector is perpendicular to $\vec{v}$ and $\vec{w}$.**

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 4 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix}\vec{i} - \begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix}\vec{j} + \begin{vmatrix} 0 & 4 \\ 0 & 1 \end{vmatrix}\vec{k} \quad \textit{Expand by minors.}$$

$$= 11\vec{i} - 0\vec{j} + 0\vec{k}$$

$$= 11\vec{i} \text{ or } \langle 11, 0, 0\rangle$$

Find the inner products.

$\langle 11, 0, 0\rangle \cdot \langle 0, 4, 1\rangle$
$11(0) + 0(4) + 0(1) = 0$

$\langle 11, 0, 0\rangle \cdot \langle 0, 1, 3\rangle$
$11(0) + 0(1) + 0(3) = 0$

Since the inner products are zero, the cross product $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$.

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8-4

Practice

Perpendicular Vectors

Find each inner product and state whether the vectors are perpendicular. Write yes or no.

1. $\langle 3, 6\rangle \cdot \langle -4, 2\rangle$

2. $\langle -1, 4\rangle \cdot \langle 3, -2\rangle$

3. $\langle 2, 0\rangle \cdot \langle -1, -1\rangle$

4. $\langle -2, 0, 1\rangle \cdot \langle 3, 2, -3\rangle$

5. $\langle -4, -1, 1\rangle \cdot \langle 1, -3, 4\rangle$

6. $\langle 0, 0, 1\rangle \cdot \langle 1, -2, 0\rangle$

Find each cross product. Then verify that the resulting vector is perpendicular to the given vectors.

7. $\langle 1, 3, 4\rangle \times \langle -1, 0, -1\rangle$

8. $\langle 3, 1, -6\rangle \times \langle -2, 4, 3\rangle$

9. $\langle 3, 1, 2\rangle \times \langle 2, -3, 1\rangle$

10. $\langle 4, -1, 0\rangle \times \langle 5, -3, -1\rangle$

11. $\langle -6, 1, 3\rangle \times \langle -2, -2, 1\rangle$

12. $\langle 0, 0, 6\rangle \times \langle 3, -2, -4\rangle$

13. ***Physics*** Janna is using a force of 100 pounds to push a cart up a ramp. The ramp is 6 feet long and is at a 30° angle with the horizontal. How much work is Janna doing in the vertical direction? (*Hint:* Use the sine ratio and the formula $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}}$.)

8-4

Enrichment

Vector Equations

Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ be fixed vectors. The equation $f(x) = \vec{\mathbf{a}} - 2x\vec{\mathbf{b}} + x^2\vec{\mathbf{c}}$ defines a vector function of x. For the values of x shown, the assigned vectors are given below.

x		–2	–1	0	1 2
***f*(*x*)**	$\vec{a} + 4\vec{b} + 4\vec{c}$	$\vec{a} + 2\vec{b} + \vec{c}$	$\vec{a}$	$\vec{a} - 2\vec{b} + \vec{c}$	$\vec{a} - 4\vec{b} + 4\vec{c}$

If $\vec{\mathbf{a}} = \langle 0, 1\rangle$, $\vec{\mathbf{b}} = \langle 1, 1\rangle$, and $\vec{\mathbf{c}} = \langle 2, -2\rangle$, the resulting vectors for the values of x are as follows.

x	–2	–1	0	1	2
***f*(*x*)**	$\langle 12, -3\rangle$	$\langle 4, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, -3\rangle$	$\langle 4, -11\rangle$

For each of the following, complete the table of resulting vectors.

1. $f(x) = x^3\vec{\mathbf{a}} - 2x^2\vec{\mathbf{b}} + 3x\vec{\mathbf{c}}$

$\vec{\mathbf{a}} = \langle 1, 1\rangle$ $\vec{\mathbf{b}} = \langle 2, 3\rangle$ $\vec{\mathbf{c}} = \langle 3, -1\rangle$

x	$f(x)$
–1	
0	
1	
2	

2. $f(x) = 2x^2\vec{\mathbf{a}} + 3x\vec{\mathbf{b}} - 5\vec{\mathbf{c}}$

$\vec{\mathbf{a}} = \langle 0, 1, 1\rangle$ $\vec{\mathbf{b}} = \langle 1, 0, 1\rangle$ $\vec{\mathbf{c}} = \langle 1, 1, 0\rangle$

x	$f(x)$
–2	
–1	
0	
1	

3. $f(x) = x^2\vec{\mathbf{c}} + 3x\vec{\mathbf{a}} - 4\vec{\mathbf{b}}$

$\vec{\mathbf{a}} = \langle 1, 1, 1\rangle$ $\vec{\mathbf{b}} = \langle 3, 2, 1\rangle$ $\vec{\mathbf{c}} = \langle 0, 1, 2\rangle$

x	$f(x)$
0	
1	
2	
3	

4. $f(x) = x^3\vec{\mathbf{a}} - x\vec{\mathbf{b}} + 3\vec{\mathbf{c}}$

$\vec{\mathbf{a}} = \langle 0, 1, -2\rangle$ $\vec{\mathbf{b}} = \langle 1, -2, 0\rangle$ $\vec{\mathbf{c}} = \langle -2, 0, 1\rangle$

x	$f(x)$
–1	
0	
1	
3	

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8-6

Study Guide

Vectors and Parametric Equations

Vector equations and **parametric equations** allow us to model movement.

Example 1 **Write a vector equation describing a line passing through $P_1(8, 4)$ and parallel to $\vec{\mathbf{a}} = \langle 6, -1\rangle$. Then write parametric equations of the line.**

Let the line ℓ through $P_1(8, 4)$ be parallel to $\vec{\mathbf{a}}$. For any point $P_2(x, y)$ on ℓ, $P_1P_2\langle x - 8, y - 4\rangle$. Since P_1P_2 is on ℓ and is parallel to $\vec{\mathbf{a}}$, $P_1P_2 = t\vec{\mathbf{a}}$, for some value t. By substitution, we have $\langle x - 8, y - 4\rangle = t\langle 6, -1\rangle$.

Therefore, the equation $\langle x - 8, y - 4\rangle = t\langle 6, -1\rangle$ is a vector equation describing all of the points (x, y) on ℓ parallel to $\vec{\mathbf{a}}$ through $P_1(8, 4)$.

Use the general form of the parametric equations of a line with $\langle a_1, a_2\rangle = \langle 6, -1\rangle$ and $\langle x_1, y_1\rangle = \langle 8, 4\rangle$.

$$x = x_1 + ta_1 \qquad y = y_1 + ta_2$$
$$x = 8 + t(6) \qquad y = 4 + t(-1)$$
$$x = 8 + 6t \qquad y = 4 - t$$

Parametric equations for the line are $x = 8 + 6t$ and $y = 4 - t$.

Example 2 **Write an equation in slope-intercept form of the line whose parametric equations are $x = -3 + 4t$ and $y = 3 + 4t$.**

Solve each parametric equation for t.

$$x = -3 + 4t \qquad y = 3 + 4t$$
$$x + 3 = 4t \qquad y - 3 = 4t$$
$$\frac{x + 3}{4} = t \qquad \frac{y - 3}{4} = t$$

Use substitution to write an equation for the line without the variable t.

$\frac{x + 3}{4} = \frac{y - 3}{4}$ *Substitute.*

$(x + 3)(4) = 4(y - 3)$ *Cross multiply.*

$4x + 12 = 4y - 12$ *Simplify.*

$y = x + 6$ *Solve for y.*

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8-6

Practice

Vectors and Parametric Equations

Write a vector equation of the line that passes through point P and is parallel to $\vec{a}$. Then write parametric equations of the line.

1. $P(-2, 1)$, $\vec{\mathbf{a}} = \langle 3, -4 \rangle$

2. $P(3, 7)$, $\vec{\mathbf{a}} = \langle 4, 5 \rangle$

3. $P(2, -4)$, $\vec{\mathbf{a}} = \langle 1, 3 \rangle$

4. $P(5, -8)$, $\vec{\mathbf{a}} = \langle 9, 2 \rangle$

Write parametric equations of the line with the given equation.

5. $y = 3x - 8$

6. $y = -x + 4$

7. $3x - 2y = 6$

8. $5x + 4y = 20$

Write an equation in slope-intercept form of the line with the given parametric equations.

9. $x = 2t + 3$
$y = t - 4$

10. $x = t + 5$
$y = -3t$

11. ***Physical Education*** Brett and Chad are playing touch football in gym class. Brett has to tag Chad before he reaches a 20-yard marker. Chad follows a path defined by $\langle x - 1, y - 19 \rangle = t\langle 0, 1 \rangle$, and Brett follows a path defined by $\langle x - 12, y - 0 \rangle = t\langle -11, 19 \rangle$. Write parametric equations for the paths of Brett and Chad. Will Brett tag Chad before he reaches the 20-yard marker?

Enrichment

Using Parametric Equations to Find the Distance from a Point to a Plane

You can use parametric equations to help you find the distance from a point not on a plane to a given plane.

Example **Find the distance from $P(-1, 1, 0)$ to the plane $x + 2y - z = 4$.**

Use the coefficients of the equation of the plane and the coordinates of the point to write the ratios below.

$$\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-0}{-1}$$

The denominators of these ratios represent a vector that is perpendicular to the plane, and passes through the given point.

Set t equal to each of the above ratios. Then, $t = \frac{x+1}{1}$, $t = \frac{y-1}{2}$, and $t = \frac{z-0}{-1}$.

So, $x = t - 1$, $y = 2t + 1$, and $z = -t$ are parametric equations of the line.

Substitute these values into the equation of the plane.
$(t - 1) + 2(2t + 1) - (-t) = 4$
Solve for t: $6t + 1 = 4$

$$t = \frac{1}{2}$$

This means that $t = \frac{1}{2}$ at the point of

intersection of the vector and the plane.
The point of intersection is $\left(\frac{1}{2} - 1, 2\left(\frac{1}{2}\right) + 1, -\frac{1}{2}\right)$

or $\left(-\frac{1}{2}, 2, -\frac{1}{2}\right)$.

Use the distance formula:

$$d = \sqrt{\left(-1 - \left(-\frac{1}{2}\right)\right)^2 + (1-2)^2 + \left(0 - \left(-\frac{1}{2}\right)\right)^2} \approx 1.2 \text{ units}$$

Find the distance from the given point to the given plane. Round your answers to the nearest tenth.

1. from $(2, 0, -1)$ to $x - 2y + z = 3$

2. from $(1, 1, -1)$ to $2x + y - 3z = 5$