96. $f(x) = \begin{cases} (x-1)^3, & x \le 1 \\ (x-1)^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{(x-1)^3 - 0}{x - 1}$$
$$= \lim_{x \to 1^-} (x-1)^2 = 0.$$

The derivative from the right is

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{(x-1)^2 - 0}{x - 1}$$
$$= \lim_{x \to 1^+} (x-1) = 0.$$

The one-sided limits are equal. Therefore, f is differentiable at $x = 1$. $\left(f'(1) = 0\right)$

97. Note that f is continuous at $x = 2$.

$$f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$$

The derivative from the left is

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{(x^2 + 1) - 5}{x - 2}$$
$$= \lim_{x \to 2^-} (x + 2) = 4.$$

The derivative from the right is

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{(4x - 3) - 5}{x - 2} = \lim_{x \to 2^+} 4 = 4.$$

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. $\left(f'(2) = 4\right)$

98. Note that f is continuous at $x = 2$.

$$f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \geq 2 \end{cases}$$

The derivative from the left is

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{\left(\frac{1}{2}x + 1\right) - 2}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{\frac{1}{2}(x - 2)}{x - 2} = \frac{1}{2}.$$

The derivative from the right is

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{\sqrt{2x} - 2}{x - 2} \cdot \frac{\sqrt{2x} + 2}{\sqrt{2x} + 2}$$

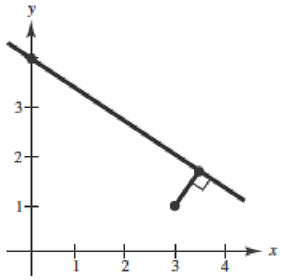
$$= \lim_{x \to 2^+} \frac{2x - 4}{(x - 2)\left(\sqrt{2x} + 2\right)}$$

$$= \lim_{x \to 2^+} \frac{2(x - 2)}{(x - 2)\left(\sqrt{2x} + 2\right)}$$

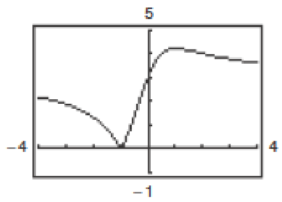
$$= \lim_{x \to 2^+} \frac{2}{\sqrt{2x} + 2} = \frac{1}{2}.$$

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. $\left(f'(2) = \frac{1}{2}\right)$

99. (a) The distance from $(3, 1)$ to the line $mx - y + 4 = 0$ is $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \dfrac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}} = \dfrac{|3m + 3|}{\sqrt{m^2 + 1}}$

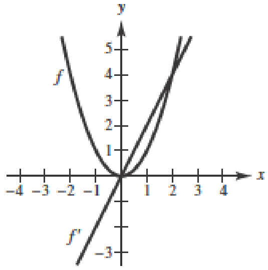


(b)

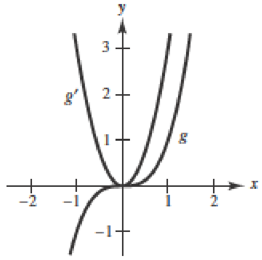


The function d is not differentiable at $m = -1$. This corresponds to the line $y = -x + 4$, which passes through the point $(3, 1)$.

100. (a) $f(x) = x^2$ and $f'(x) = 2x$



(b) $g(x) = x^3$ and $g'(x) = 3x^2$



(c) The derivative is a polynomial of degree 1 less than the original function. If $h(x) = x^n$, then $h'(x) = nx^{n-1}$.

(d) If $f(x) = x^4$, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^4 - x^4}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 - x^4}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x\left(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3\right)}{\Delta x} = \lim_{\Delta x \to 0}\left(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3\right) = 4x^3.$$

So, if $f(x) = x^4$, then $f'(x) = 4x^3$ which is consistent with the conjecture. However, this is not a proof because you must verify the conjecture for all integer values of n, $n \geq 2$.

101. False. The slope is $\lim_{\Delta x \to 0} \dfrac{f(2 + \Delta x) - f(2)}{\Delta x}$.

102. False. $y = |x - 2|$ is continuous at $x = 2$, but is not differentiable at $x = 2$. (Sharp turn in the graph)

103. False. If the derivative from the left of a point does not equal the derivative from the right of a point, then the derivative does not exist at that point. For example, if $f(x) = |x|$, then the derivative from the left at $x = 0$ is -1 and the derivative from the right at $x = 0$ is 1. At $x = 0$, the derivative does not exist.

104. True—see Theorem 2.1.

105. $f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Using the Squeeze Theorem, you have $-|x| \le x\sin(1/x) \le |x|$, $x \neq 0$. So, $\lim_{x\to 0} x\sin(1/x) = 0 = f(0)$ and f is continuous at $x = 0$. Using the alternative form of the derivative, you have

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{x\sin(1/x) - 0}{x - 0} = \lim_{x\to 0}\left(\sin\frac{1}{x}\right).$$

Because this limit does not exist ($\sin(1/x)$ oscillates between -1 and 1), the function is not differentiable at $x = 0$.

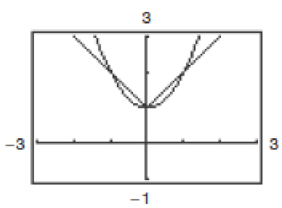
$g(x) = \begin{cases} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Using the Squeeze Theorem again, you have $-x^2 \le x^2\sin(1/x) \le x^2$, $x \neq 0$. So, $\lim_{x\to 0} x^2\sin(1/x) = 0 = g(0)$ and g is continuous at $x = 0$. Using the alternative form of the derivative again, you have

$$\lim_{x\to 0}\frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0}\frac{x^2\sin(1/x) - 0}{x - 0} = \lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

Therefore, g is differentiable at $x = 0$, $g'(0) = 0$.

106.



As you zoom in, the graph of $y_1 = x^2 + 1$ appears to be locally the graph of a horizontal line, whereas the graph of $y_2 = |x| + 1$ always has a sharp corner at $(0, 1)$. y_2 is not differentiable at $(0, 1)$.