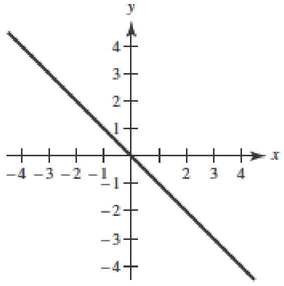
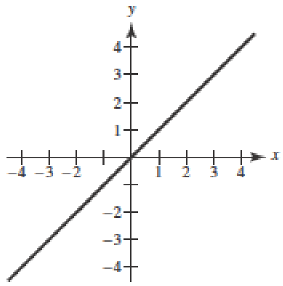
51. Answers will vary.

Sample answer: $y = -x$



52. Answers will vary.

Sample answer: $y = x$



53. $f(x) = 5 - 3x$ and $c = 1$

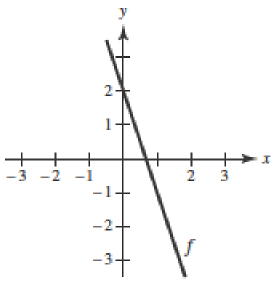
54. $f(x) = x^3$ and $c = -2$

55. $f(x) = -x^2$ and $c = 6$

56. $f(x) = 2\sqrt{x}$ and $c = 9$

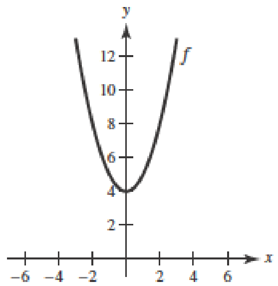
57. $f(0) = 2$ and $f'(x) = -3, -\infty < x < \infty$

$f(x) = -3x + 2$



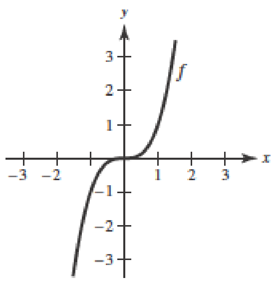
58. $f(0) = 4,\ f'(0) = 0;\ f'(x) < 0$ for $x < 0,\ f'(x) > 0$ for $x > 0$

Answers will vary: *Sample answer:* $f(x) = x^2 + 4$



59. $f(0) = 0;\ f'(0) = 0;\ f'(x) > 0$ if $x \neq 0$

Answers will vary: *Sample answer:* $f(x) = x^3$



60. (a) If $f'(c) = 3$ and f is odd, then $f'(-c) = f'(c) = 3$.

(b) If $f'(c) = 3$ and f is even, then

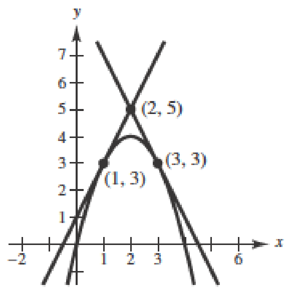
$$f'(-c) = -f'(c) = -3.$$

61. Let (x_0, y_0) be a point of tangency on the graph of f. By the limit definition for the derivative, $f'(x) = 4 - 2x$. The slope of the line through $(2, 5)$ and (x_0, y_0) equals the derivative of f at x_0:

$$\frac{5 - y_0}{2 - x_0} = 4 - 2x_0$$
$$5 - y_0 = (2 - x_0)(4 - 2x_0)$$
$$5 - (4x_0 - x_0^2) = 8 - 8x_0 + 2x_0^2$$
$$0 = x_0^2 - 4x_0 + 3$$
$$0 = (x_0 - 1)(x_0 - 3) \Rightarrow x_0 = 1, 3$$

Therefore, the points of tangency are $(1, 3)$ and $(3, 3)$, and the corresponding slopes are 2 and -2. The equations of the tangent lines are:

$$y - 5 = 2(x - 2) \qquad y - 5 = -2(x - 2)$$
$$y = 2x + 1 \qquad y = -2x + 9$$



62. Let (x_0, y_0) be a point of tangency on the graph of *f*. By the limit definition for the derivative, $f'(x) = 2x$. The slope of the line through $(1, -3)$ and (x_0, y_0) equals the derivative of *f* at x_0:

$$\frac{-3 - y_0}{1 - x_0} = 2x_0$$

$$-3 - y_0 = (1 - x_0)2x_0$$

$$-3 - x_0^2 = 2x_0 - 2x_0^2$$

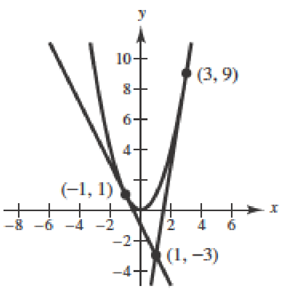
$$x_0^2 - 2x_0 - 3 = 0$$

$$(x_0 - 3)(x_0 + 1) = 0 \Rightarrow x_0 = 3, -1$$

Therefore, the points of tangency are $(3, 9)$ and $(-1, 1)$, and the corresponding slopes are 6 and -2. The equations of the tangent lines are:

$$y + 3 = 6(x - 1) \qquad y + 3 = -2(x - 1)$$

$$y = 6x - 9 \qquad y = -2x - 1$$



63. (a) $f(x) = x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

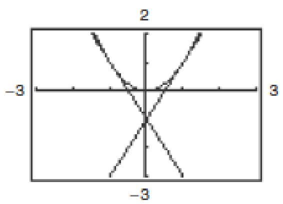
At $x = -1$, $f'(-1) = -2$ and the tangent line is

$y - 1 = -2(x + 1)$ or $y = -2x - 1$.

At $x = 0$, $f'(0) = 0$ and the tangent line is $y = 0$.

At $x = 1$, $f'(1) = 2$ and the tangent line is

$y = 2x - 1$.



For this function, the slopes of the tangent lines are always distinct for different values of x.

(b) $g'(x) = \lim_{\Delta x \to 0} \dfrac{g(x + \Delta x) - g(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{(x + \Delta x)^3 - x^3}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{\Delta x\left(3x^2 + 3x(\Delta x) + (\Delta x)^2\right)}{\Delta x}$

$= \lim_{\Delta x \to 0}\left(3x^2 + 3x(\Delta x) + (\Delta x)^2\right) = 3x^2$

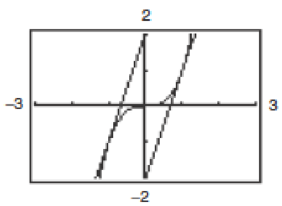
At $x = -1$, $g'(-1) = 3$ and the tangent line is

$y + 1 = 3(x + 1)$ or $y = 3x + 2$.

At $x = 0$, $g'(0) = 0$ and the tangent line is $y = 0$.

At $x = 1$, $g'(1) = 3$ and the tangent line is

$y - 1 = 3(x - 1)$ or $y = 3x - 2$.



For this function, the slopes of the tangent lines are sometimes the same.

64. (a) $g'(0) = -3$

(b) $g'(3) = 0$

(c) Because $g'(1) = -\frac{8}{3}$, g is decreasing (falling) at $x = 1$.

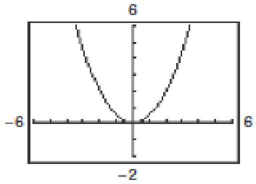
(d) Because $g'(-4) = \frac{7}{3}$, g is increasing (rising) at $x = -4$.

(e) Because $g'(4)$ and $g'(6)$ are both positive, $g(6)$ is greater than $g(4)$, and $g(6) - g(4) > 0$.

(f) No, it is not possible. All you can say is that g is decreasing (falling) at $x = 2$.

65. $f(x) = \frac{1}{2}x^2$

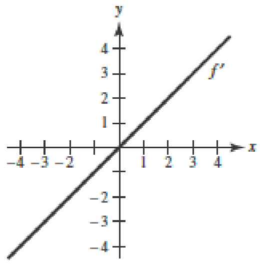
(a)



$f'(0) = 0, f'(1/2) = 1/2, f'(1) = 1, f'(2) = 2$

(b) By symmetry: $f'(-1/2) = -1/2, f'(-1) = -1, f'(-2) = -2$

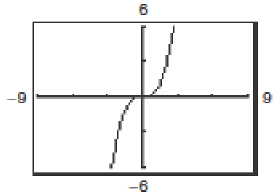
(c)



(d) $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{2}(x + \Delta x)^2 - \frac{1}{2}x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{2}\left(x^2 + 2x(\Delta x) + (\Delta x)^2\right) - \frac{1}{2}x^2}{\Delta x} = \lim_{\Delta x \to 0}\left(x + \frac{\Delta x}{2}\right) = x$

66. $f(x) = \frac{1}{3}x^3$

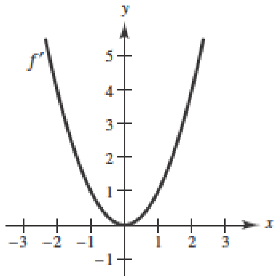
(a)



$f'(0) = 0, f'(1/2) = 1/4, f'(1) = 1, f'(2) = 4, f'(3) = 9$

(b) By symmetry: $f'(-1/2) = 1/4, f'(-1) = 1, \ f'(-2) = 4, f'(-3) = 9$

(c)



(d) $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{3}(x + \Delta x)^3 - \frac{1}{3}x^3}{\Delta x}$$

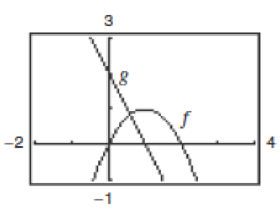
$$= \lim_{\Delta x \to 0} \frac{\frac{1}{3}\left(x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3\right) - \frac{1}{3}x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left[x^2 + x(\Delta x) + \frac{1}{3}(\Delta x)^2\right] = x^2$$

67. $$g(x) = \frac{f(x + 0.01) - f(x)}{0.01}$$

$$= \left[2(x + 0.01) - (x + 0.01)^2 - 2x + x^2\right]100$$

$$= 2 - 2x - 0.01$$

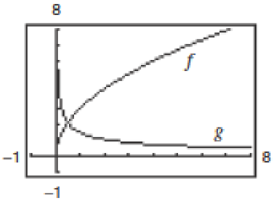


The graph of $g(x)$ is approximately the graph of

$f'(x) = 2 - 2x.$

68. $g(x) = \dfrac{f(x + 0.01) - f(x)}{0.01}$

$= \left(3\sqrt{x + 0.01} - 3\sqrt{x}\right)100$



The graph of $g(x)$ is approximately the graph of

$f'(x) = \dfrac{3}{2\sqrt{x}}.$

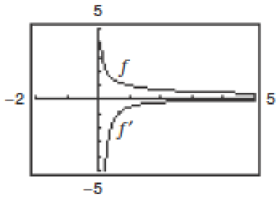
69. $f(2) = 2(4 - 2) = 4,\ f(2.1) = 2.1(4 - 2.1) = 3.99$

$f'(2) \approx \dfrac{3.99 - 4}{2.1 - 2} = -0.1 \quad \left[\text{Exact: } f'(2) = 0\right]$

70. $f(2) = \dfrac{1}{4}\left(2^3\right) = 2,\ f(2.1) = 2.31525$

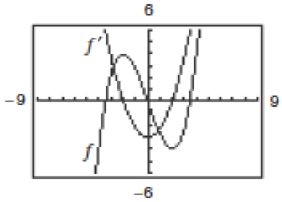
$f'(2) \approx \dfrac{2.31525 - 2}{2.1 - 2} = 3.1525\left[\text{Exact: } f'(2) = 3\right]$

71. $f(x) = \dfrac{1}{\sqrt{x}}$ and $f'(x) = \dfrac{-1}{2x^{3/2}}.$



As $x \to \infty$, f is nearly horizontal and thus $f' \approx 0$.

72. $f(x) = \dfrac{x^3}{4} - 3x$ and $f'(x) = \dfrac{3}{4}x^2 - 3$



When f is increasing, $f' > 0$.

When f is decreasing, $f' < 0$.

73. $f(x) = x^2 - 5, c = 3$

$$f'(3) = \lim_{x\to 3}\frac{f(x) - f(3)}{x - 3}$$
$$= \lim_{x\to 3}\frac{x^2 - 5 - (9 - 5)}{x - 3}$$
$$= \lim_{x\to 3}\frac{(x - 3)(x + 3)}{x - 3}$$
$$= \lim_{x\to 3}(x + 3) = 6$$

74. $g(x) = x(x - 1) = x^2 - x, c = 1$

$$g'(1) = \lim_{x\to 1}\frac{g(x) - g(1)}{x - 1}$$
$$= \lim_{x\to 1}\frac{x^2 - x - 0}{x - 1}$$
$$= \lim_{x\to 1}\frac{x(x - 1)}{x - 1} = \lim_{x\to 1} x = 1$$

75. $f(x) = x^3 + 6x, c = 2$

$$f'(2) = \lim_{x\to 2}\frac{f(x) - f(2)}{x - 2}$$
$$= \lim_{x\to 2}\frac{(x^3 + 6x) - 20}{x - 2}$$
$$= \lim_{x\to 2}\frac{(x - 2)(x^2 + 2x + 10)}{x - 2}$$
$$= \lim_{x\to 2}(x^2 + 2x + 10) = 18$$

76. $f(x) = x^3 + 2x^2 + 1, c = -2$

$$f'(-2) = \lim_{x\to -2}\frac{f(x) - f(-2)}{x + 2}$$
$$= \lim_{x\to -2}\frac{(x^3 + 2x^2 + 1) - 1}{x + 2}$$
$$= \lim_{x\to -2}\frac{x^2(x + 2)}{x + 2} = \lim_{x\to -2} x^2 = 4$$

77. $f(x) = \frac{2}{x}, c = 5$

$$f'(5) = \lim_{x\to 5} \frac{f(x) - f(5)}{x - 5}$$

$$= \lim_{x\to 5} \frac{\frac{2}{x} - \frac{2}{5}}{x - 5}$$

$$= \lim_{x\to 5} \frac{2(5 - x)}{5x(x - 5)}$$

$$= \lim_{x\to 5} -\frac{2}{5x} = -\frac{2}{25}$$

78. $g(x) = \sqrt{|x|}, c = 0$

$g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{\sqrt{|x|}}{x}$. Does not exist.

As $x \to 0^-, \frac{\sqrt{|x|}}{x} = \frac{-1}{\sqrt{|x|}} \to -\infty$.

As $x \to 0^+, \frac{\sqrt{|x|}}{x} = \frac{1}{\sqrt{x}} \to \infty$.

Therefore $g(x)$ is not differentiable at $x = 0$.

79. $g(x) = (x + 3)^{1/3}, c = -3$

$$g'(-3) = \lim_{x\to -3} \frac{g(x) - g(-3)}{x - (-3)}$$

$$= \lim_{x\to -3} \frac{(x + 3)^{1/3} - 0}{x + 3} = \lim_{x\to -3} \frac{1}{(x + 3)^{2/3}}.$$

Does not exist.

Therefore $g(x)$ is not differentiable at $x = -3$.

80. $f(x) = (x - 6)^{2/3}, c = 6$

$$f'(6) = \lim_{x\to 6} \frac{f(x) - f(6)}{x - 6}$$

$$= \lim_{x\to 6} \frac{(x - 6)^{2/3} - 0}{x - 6} = \lim_{x\to 6} \frac{1}{(x - 6)^{1/3}}.$$

Does not exist.

Therefore $f(x)$ is not differentiable at $x = 6$.

81. $f(x) = |x - 6|, c = 6$

$$f'(6) = \lim_{x\to 6}\frac{f(x) - f(6)}{x - 6}$$

$$= \lim_{x\to 6}\frac{|x - 6| - 0}{x - 6} = \lim_{x\to 6}\frac{|x - 6|}{x - 6}.$$

Does not exist.

Therefore $f(x)$ is not differentiable at $x = 6$.

82. $h(x) = |x + 7|, c = -7$

$$h'(-7) = \lim_{x\to -7}\frac{h(x) - h(-7)}{x - (-7)}$$

$$= \lim_{x\to -7}\frac{|x + 7| - 0}{x + 7} = \lim_{x\to -7}\frac{|x + 7|}{x + 7}.$$

Does not exist.

Therefore $h(x)$ is not differentiable at $x = -7$.

83. $f(x)$ is differentiable everywhere except at $x = 3$. (Discontinuity)

84. $f(x)$ is differentiable everywhere except at $x = \pm 3$. (Sharp turns in the graph)

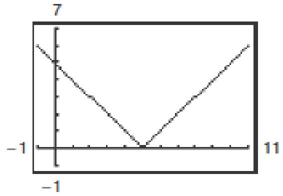
85. $f(x)$ is differentiable everywhere except at $x = -4$. (Sharp turn in the graph)

86. $f(x)$ is differentiable everywhere except at $x = \pm 2$. (Discontinuities)

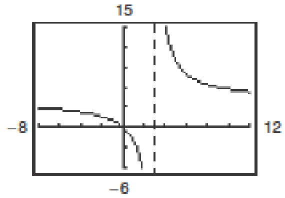
87. $f(x)$ is differentiable on the interval $(1, \infty)$. (At $x = 1$ the tangent line is vertical.)

88. $f(x)$ is differentiable everywhere except at $x = 0$. (Discontinuity)

89. $f(x) = |x - 5|$ is differentiable everywhere except at $x = -5$. There is a sharp corner at $x = 5$.

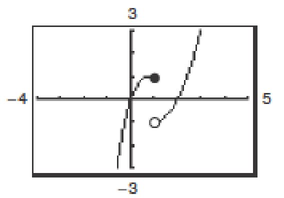


90. $f(x) = \dfrac{4x}{x - 3}$ is differentiable everywhere except at $x = 3$. f is not defined at $x = 3$. (Vertical asymptote)

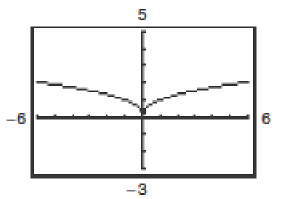


91. f is differentiable for all $x \neq 1$.

f is not continuous at $x = 1$.



92. $f(x) = x^{2/5}$ is differentiable for all $x \neq 0$. There is a sharp corner at $x = 0$.



93. $f(x) = |x - 1|$

The derivative from the left is

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{|x - 1| - 0}{x - 1} = -1.$$

The derivative from the right is

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{|x - 1| - 0}{x - 1} = 1.$$

The one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

94. $f(x) = \sqrt{1 - x^2}$

The derivative from the left does not exist because

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{\sqrt{1 - x^2} - 0}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{\sqrt{1 - x^2}}{x - 1} \cdot \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}}$$

$$= \lim_{x \to 1^-} -\frac{1 + x}{\sqrt{1 - x^2}} = -\infty.$$

(Vertical tangent)

The limit from the right does not exist since f is undefined for $x > 1$. Therefore, f is not differentiable at $x = 1$.

95. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x - 1}{x - 1} = \lim_{x \to 1^-} 1 = 1.$$

The derivative from the right is

$$\lim_{x \to 1^+} = \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^+} (x + 1) = 2.$$

The one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.