101. False. A rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n, respectively. It can have, at most, n discontinuities.

102. False. $f(1)$ is not defined and $\lim_{x\to 1} f(x)$ does not exist.

103. $\lim_{t\to 4^-} f(t) \approx 28$

$\lim_{t\to 4^+} f(t) \approx 56$

At the end of day 3, the amount of chlorine in the pool has decreased to about 28 oz. At the beginning of day 4, more chlorine was added, and the amount is now about 56 oz.

104. The functions agree for integer values of x:

$$\left.\begin{aligned} g(x) &= 3 - [\![-x]\!] = 3 - (-x) = 3 + x \\ f(x) &= 3 + [\![x]\!] = 3 + x \end{aligned}\right\} \text{for } x \text{ an integer}$$

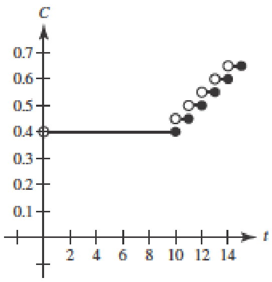
However, for non-integer values of x, the functions differ by 1.

$$f(x) = 3 + [\![x]\!] = g(x) - 1 = 2 - [\![-x]\!].$$

For example,

$$f\left(\tfrac{1}{2}\right) = 3 + 0 = 3,\ g\left(\tfrac{1}{2}\right) = 3 - (-1) = 4.$$

105. $$C(t) = \begin{cases} 0.40, & 0 < t \le 10 \\ 0.40 + 0.05[\![t - 9]\!], & t > 10,\ t \text{ not an integer} \\ 0.40 + 0.05(t - 10), & t > 10,\ t \text{ an integer} \end{cases}$$



There is a nonremovable discontinuity at each integer greater than or equal to 10.

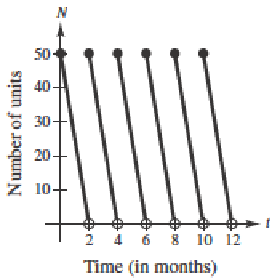
Note: You could also express C as

$$C(t) = \begin{cases} 0.40, & 0 < t \le 10 \\ 0.40 - 0.05[\![10 - t]\!], & t > 10 \end{cases}$$

106. $N(t) = 25\left(2\left[\!\left[\frac{t+2}{2}\right]\!\right] - t\right)$

t	0	1	1.8	2	3	3.8
$N(t)$	50	25	5	50	25	5

Discontinuous at every positive even integer. The company replenishes its inventory every two months.



107. Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k$, $r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.),
$f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:00 A.M.), $f(10) = s(10) - r(10) > 0$.

Because $f(0) < 0$ and $f(10) > 0$, then there must be a value t in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time t, where $0 \le t \le 10$, the position functions for the run up and the run down are equal.

108. Let $V = \frac{4}{3}\pi r^3$ be the volume of a sphere with radius r.

V is continuous on $[5, 8]$. $V(5) = \frac{500\pi}{3} \approx 523.6$ and

$V(8) = \frac{2048\pi}{3} \approx 2144.7$. Because

$523.6 < 1500 < 2144.7$, the Intermediate Value Theorem guarantees that there is at least one value r between 5 and 8 such that $V(r) = 1500$. (In fact, $r \approx 7.1012$.)

109. Suppose there exists x_1 in $[a, b]$ such that $f(x_1) > 0$ and there exists x_2 in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of x in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). So, f would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

110. Let c be any real number. Then $\lim_{x \to c} f(x)$ does not exist because there are both rational and irrational numbers arbitrarily close to c. Therefore, f is not continuous at c.

111. If $x = 0$, then $f(0) = 0$ and $\lim_{x \to 0} f(x) = 0$. So, f is continuous at $x = 0$.

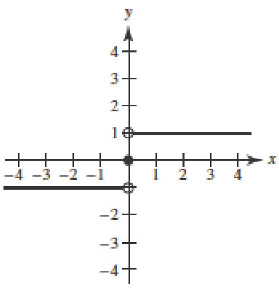
If $x \neq 0$, then $\lim_{t \to x} f(t) = 0$ for x rational, whereas $\lim_{t \to x} f(t) = \lim_{t \to x} kt = kx \neq 0$ for x irrational. So, f is not continuous for all $x \neq 0$.

112. $\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$

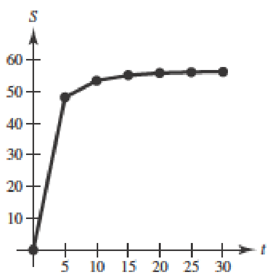
(a) $\lim_{x \to 0^-} \operatorname{sgn}(x) = -1$

(b) $\lim_{x \to 0^+} \operatorname{sgn}(x) = 1$

(c) $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist.

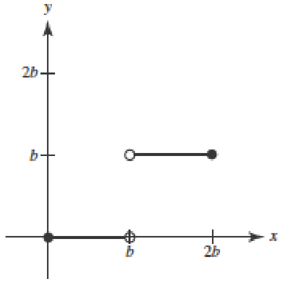


113. (a)



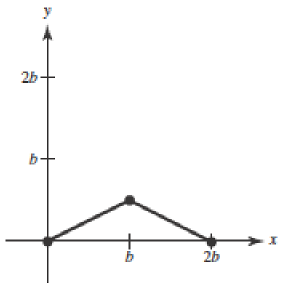
(b) There appears to be a limiting speed and a possible cause is air resistance.

114. (a) $f(x) = \begin{cases} 0, & 0 \le x < b \\ b, & b < x \le 2b \end{cases}$



NOT continuous at $x = b$.

(b) $g(x) = \begin{cases} \dfrac{x}{2}, & 0 \le x \le b \\ b - \dfrac{x}{2}, & b < x \le 2b \end{cases}$



Continuous on $[0, 2b]$.

115. $f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$

f is continuous for $x < c$ and for $x > c$. At $x = c$, you need $1 - c^2 = c$. Solving $c^2 + c - 1$, you obtain

$$c = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

116. Let y be a real number. If $y = 0$, then $x = 0$. If $y > 0$, then let $0 < x_0 < \pi/2$ such that $M = \tan x_0 > y$ (this is possible since the tangent function increases without bound on $[0, \pi/2)$). By the Intermediate Value Theorem, $f(x) = \tan x$ is continuous on $[0, x_0]$ and $0 < y < M$, which implies that there exists x between 0 and x_0 such that $\tan x = y$. The argument is similar if $y < 0$.

117. $f(x) = \dfrac{\sqrt{x + c^2} - c}{x}, c > 0$

Domain: $x + c^2 \geq 0 \Rightarrow x \geq -c^2$ and $x \neq 0, \left[-c^2, 0\right) \cup (0, \infty)$

$$\lim_{x \to 0} \frac{\sqrt{x + c^2} - c}{x} = \lim_{x \to 0} \frac{\sqrt{x + c^2} - c}{x} \cdot \frac{\sqrt{x + c^2} + c}{\sqrt{x + c^2} + c} = \lim_{x \to 0} \frac{\left(x + c^2\right) - c^2}{x\left[\sqrt{x + c^2} + c\right]} = \lim_{x \to 0} \frac{1}{\sqrt{x + c^2} + c} = \frac{1}{2c}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

118. **1.** $f(c)$ is defined.

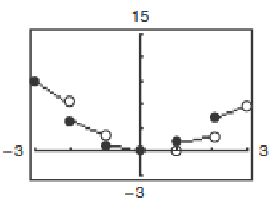
2. $\lim_{x \to c} f(x) = \lim_{\Delta x \to 0} f(c + \Delta x) = f(c)$ exists.

[Let $x = c + \Delta x$. As $x \to c, \Delta x \to 0$]

3. $\lim_{x \to c} f(x) = f(c)$.

Therefore, f is continuous at $x = c$.

119. $h(x) = x[\![x]\!]$



h has nonremovable discontinuities at $x = \pm 1, \pm 2, \pm 3, \ldots$.

120. (a) Define $f(x) = f_2(x) - f_1(x)$. Because f_1 and f_2 are continuous on $[a, b]$, so is f.

$f(a) = f_2(a) - f_1(a) > 0$ and

$f(b) = f_2(b) - f_1(b) < 0$

By the Intermediate Value Theorem, there exists c in $[a, b]$ such that $f(c) = 0$.

$f(c) = f_2(c) - f_1(c) = 0 \Rightarrow f_1(c) = f_2(c)$

(b) Let $f_1(x) = x$ and $f_2(x) = \cos x$, continuous on $[0, \pi/2]$, $f_1(0) < f_2(0)$ and $f_1(\pi/2) > f_2(\pi/2)$.

So by part (a), there exists c in $[0, \pi/2]$ such that

$c = \cos(c)$.

Using a graphing utility, $c \approx 0.739$.

121. The statement is true.

If $y \geq 0$ and $y \leq 1$, then $y(y-1) \leq 0 \leq x^2$, as desired. So assume $y > 1$. There are now two cases.

Case 1: If $x \leq y - \frac{1}{2}$, then $2x + 1 \leq 2y$ and

$$\begin{aligned} y(y-1) &= y(y+1) - 2y \\ &\leq (x+1)^2 - 2y \\ &= x^2 + 2x + 1 - 2y \\ &\leq x^2 + 2y - 2y \\ &= x^2 \end{aligned}$$

Case 2: If $x \geq y - \frac{1}{2}$

$$\begin{aligned} x^2 &\geq \left(y - \tfrac{1}{2}\right)^2 \\ &= y^2 - y + \tfrac{1}{4} \\ &> y^2 - y \\ &= y(y-1) \end{aligned}$$

In both cases, $y(y-1) \leq x^2$.

122. $P(1) = P(0^2 + 1) = P(0)^2 + 1 = 1$

$P(2) = P(1^2 + 1) = P(1)^2 + 1 = 2$

$P(5) = P(2^2 + 1) = P(2)^2 + 1 = 5$

Continuing this pattern, you see that $P(x) = x$ for infinitely many values of x. So, the finite degree polynomial must be constant: $P(x) = x$ for all x.