51. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1$

2. $\left.\begin{array}{l} \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x = 1 \\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} x^2 = 1 \end{array}\right\} \lim\limits_{x\to 1} f(x) = 1$

3. $f(-1) = \lim\limits_{x\to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

52. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1^2 = 1$

2. $\left.\begin{array}{l} \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (-2x + 3) = 1 \\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} x^2 = 1 \end{array}\right\} \lim\limits_{x\to 1} f(x) = 1$

3. $f(1) = \lim\limits_{x\to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

53. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

1. $f(2) = -2(2) = -4$

2. $\left.\begin{array}{l} \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (-2x) = -4 \\ \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} \left(x^2 - 4x + 1\right) = -3 \end{array}\right\} \lim\limits_{x\to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

54. $f(x) = \begin{cases} \dfrac{x}{2} + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

1. $f(2) = \dfrac{2}{2} + 1 = 2$

2. $\left.\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-}\left(\frac{x}{2} + 1\right) = 2 \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+}(3 - x) = 1 \end{aligned}\right\} \lim_{x \to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

55. $f(x) = \begin{cases} \tan\dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

$= \begin{cases} \tan\dfrac{\pi x}{4}, & -1 < x < 1 \\ x, & x \le -1 \text{ or } x \ge 1 \end{cases}$

has **possible** discontinuities at $x = -1$, $x = 1$.

1. $f(-1) = -1$ $\qquad f(1) = 1$

2. $\lim_{x \to -1} f(x) = -1$ $\qquad \lim_{x \to 1} f(x) = 1$

3. $f(-1) = \lim_{x \to -1} f(x)$ $\qquad f(1) = \lim_{x \to 1} f(x)$

f is continuous at $x = \pm 1$, therefore, f is continuous for all real x.

56. $f(x) = \begin{cases} \csc\dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

$= \begin{cases} \csc\dfrac{\pi x}{6}, & 1 \le x \le 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$

has **possible** discontinuities at $x = 1$, $x = 5$.

1. $f(1) = \csc\dfrac{\pi}{6} = 2$ $\qquad f(5) = \csc\dfrac{5\pi}{6} = 2$

2. $\lim_{x \to 1} f(x) = 2$ $\qquad \lim_{x \to 5} f(x) = 2$

3. $f(1) = \lim_{x \to 1} f(x)$ $\qquad f(5) = \lim_{x \to 5} f(x)$

f is continuous at $x = 1$ and $x = 5$, therefore, f is continuous for all real x.

57. $f(x) = \tan\frac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, k is an integer.

58. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

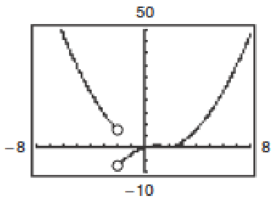
59. $f(x) = 5 - [\![x]\!]$ has nonremovable discontinuities at each integer k.

60. $f(x) = [\![x - 8]\!]$ has nonremovable discontinuities at each integer k.

61. $\lim_{x\to 0^+} f(x) = 0$

$\lim_{x\to 0^-} f(x) = 0$

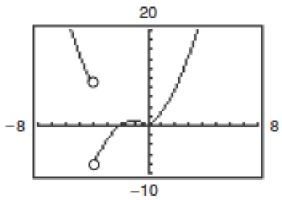
f is not continuous at $x = -2$.



62. $\lim_{x\to 0^+} f(x) = 0$

$\lim_{x\to 0^-} f(x) = 0$

f is not continuous at $x = -4$.



63. $f(1) = 3$

Find a so that $\lim_{x\to 1^-}(ax - 4) = 3$

$$a(1) - 4 = 3$$
$$a = 7.$$

64. $f(1) = 3$

Find a so that $\lim_{x\to 1^+}(ax + 5) = 3$

$$a(1) + 5 = 3$$
$$a = -2.$$

65. $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{4\sin x}{x} = 4$

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (a - 2x) = a$

Let $a = 4$.

66. $f(2) = 8$

Find a so that $\lim_{x\to 2^+} ax^2 = 8 \Rightarrow a = \frac{8}{2^2} = 2.$

67. $\lim_{x\to a} g(x) = \lim_{x\to a} \frac{x^2 - a^2}{x - a}$

$= \lim_{x\to a}(x + a) = 2a$

Find a such $2a = 8 \Rightarrow a = 4$.

68. Find a and b such that $\lim_{x\to -1^+}(ax + b) = -a + b = 2$ and $\lim_{x\to 3^-}(ax + b) = 3a + b = -2$.

$$\begin{aligned} a - b &= -2 \\ (+)3a + b &= -2 \\ \hline 4a \quad &= -4 \\ a &= -1 \\ b &= 2 + (-1) = 1 \end{aligned}$$

$$f(x) = \begin{cases} 2, & x \le -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$$

69. $f(g(x)) = (x - 1)^2$

Continuous for all real x.

70. $f(g(x)) = \frac{1}{\sqrt{x - 1}}$

Nonremovable discontinuity at $x = 1$. Continuous for all $x > 1$.

71. $f(g(x)) = \frac{1}{(x^2 + 5) - 6} = \frac{1}{x^2 - 1}$

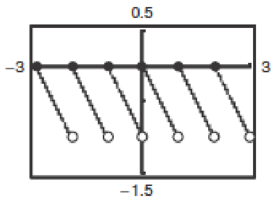
Nonremovable discontinuities at $x = \pm 1$

72. $f(g(x)) = \sin x^2$

Continuous for all real x

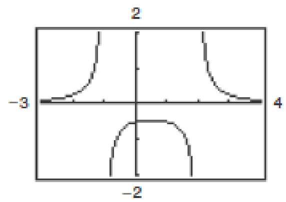
73. $y = [\![x]\!] - x$

Nonremovable discontinuity at each integer



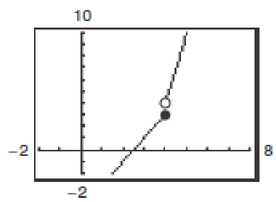
74. $h(x) = \dfrac{1}{(x+1)(x-2)}$

Nonremovable discontinuities at $x = -1$ and $x = 2$.



75. $g(x) = \begin{cases} x^2 - 3x, & x > 4 \\ 2x - 5, & x \le 4 \end{cases}$

There is a nonremovable discontinuity at $x = 4$.



76. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

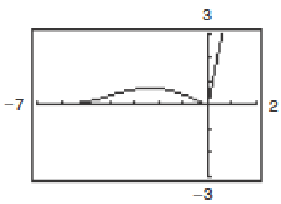
$f(0) = 5(0) = 0$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \dfrac{(\cos x - 1)}{x} = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (5x) = 0$

Therefore, $\lim_{x \to 0} f(x) = 0 = f(0)$ and f is continuous on the entire real line.

($x = 0$ was the only possible discontinuity.)



77. $f(x) = \dfrac{x}{x^2 + x + 2}$

Continuous on $(-\infty, \infty)$

78. $f(x) = x\sqrt{x + 3}$

Continuous on $[-3, \infty)$

79. $f(x) = \sec \dfrac{\pi x}{4}$

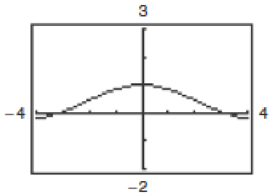
Continuous on:

$\ldots, (-6, -2), (-2, 2), (2, 6), (6, 10), \ldots$

80. $f(x) = \dfrac{x + 1}{\sqrt{x}}$

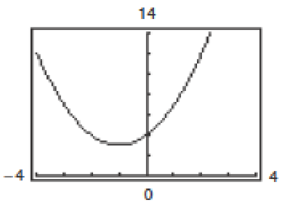
Continuous on $(0, \infty)$

81. $f(x) = \dfrac{\sin x}{x}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Because $f(0)$ is not defined, you know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

82. $f(x) = \dfrac{x^3 - 8}{x - 2}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Because $f(2)$ is not defined, you know that f has a discontinuity at $x = 2$. This discontinuity is removable so it does not show up on the graph.

83. $f(x) = \frac{1}{12}x^4 - x^3 + 4$ is continuous on the interval $[1, 2]$. $f(1) = \frac{37}{12}$ and $f(2) = -\frac{8}{3}$. By the Intermediate Value Theorem, there exists a number c in $[1, 2]$ such that $f(c) = 0$.

84. $f(x) = x^3 + 5x - 3$ is continuous on the interval $[0, 1]$. $f(0) = -3$ and $f(1) = 3$. By the Intermediate Value Theorem, there exists a number c in $[0, 1]$ such that $f(c) = 0$.

85. $f(x) = x^2 - 2 - \cos x$ is continuous on $[0, \pi]$. $f(0) = -3$ and $f(\pi) = \pi^2 - 1 \approx 8.87 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and π.

86. $f(x) = -\frac{5}{x} + \tan\left(\frac{\pi x}{10}\right)$ is continuous on the interval $[1, 4]$.

$f(1) = -5 + \tan\left(\frac{\pi}{10}\right) \approx -4.7$ and

$f(4) = -\frac{5}{4} + \tan\left(\frac{2\pi}{5}\right) \approx 1.8$. By the Intermediate Value Theorem, there exists a number c in $[1, 4]$ such that $f(c) = 0$.

87. $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.68$. Using the root feature, you find that $x \approx 0.6823$.

88. $f(x) = x^3 + 5x - 3$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -3$ and $f(1) = 3$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.56$. Using the root feature, you find that $x \approx 0.5641$.

89. $g(t) = 2 \cos t - 3t$

g is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $g(t)$, you find that $t \approx 0.56$. Using the root feature, you find that $t \approx 0.5636$.

90. $h(\theta) = 1 + \theta - 3\tan\theta$

h is continuous on $[0, 1]$.

$h(0) = 1 > 0$ and $h(1) \approx -2.67 < 0$.

By the Intermediate Value Theorem, $h(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $h(\theta)$, you find that $\theta \approx 0.45$. Using the root feature, you find that $\theta \approx 0.4503$.

91. $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$-1 < 11 < 29$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x = -4 \text{ or } x = 3$$

$c = 3$ ($x = -4$ is not in the interval.)

So, $f(3) = 11$.

92. $f(x) = x^2 - 6x + 8$

f is continuous on $[0, 3]$.

$f(0) = 8$ and $f(3) = -1$

$-1 < 0 < 8$

The Intermediate Value Theorem applies.

$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$
$$x = 2 \text{ or } x = 4$$

$c = 2$ ($x = 4$ is not in the interval.)

So, $f(2) = 0$.

93. $f(x) = x^3 - x^2 + x - 2$

f is continuous on $[0, 3]$.

$f(0) = -2$ and $f(3) = 19$

$-2 < 4 < 19$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$
$$x^3 - x^2 + x - 6 = 0$$
$$(x - 2)(x^2 + x + 3) = 0$$
$$x = 2$$

$(x^2 + x + 3$ has no real solution.$)$

$$c = 2$$

So, $f(2) = 4$.

94. $f(x) = \dfrac{x^2 + x}{x - 1}$

f is continuous on $\left[\frac{5}{2}, 4\right]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$f\left(\frac{5}{2}\right) = \frac{35}{6}$ and $f(4) = \frac{20}{3}$

$\frac{35}{6} < 6 < \frac{20}{3}$

The Intermediate Value Theorem applies.

$$\frac{x^2 + x}{x - 1} = 6$$
$$x^2 + x = 6x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2 \text{ or } x = 3$$

$c = 3$ $(x = 2$ is not in the interval.$)$

So, $f(3) = 6$.

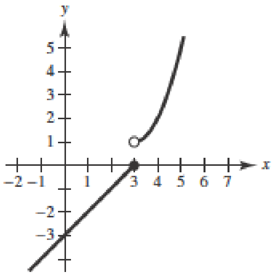
95. (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

96. Answers will vary. *Sample answer*:



The function is not continuous at $x = 3$ because

$\lim_{x \to 3^+} f(x) = 1 \neq 0 = \lim_{x \to 3^-} f(x)$.

97. If f and g are continuous for all real x, then so is $f + g$ (Theorem 1.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x, but f/g is not continuous at $x = \pm 1$.

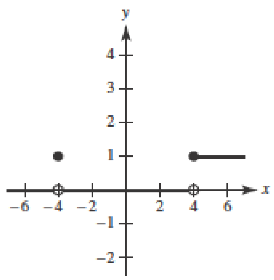
98. A discontinuity at c is removable if the function f can be made continuous at c by appropriately defining (or redefining) $f(c)$. Otherwise, the discontinuity is nonremovable.

(a) $f(x) = \frac{|x - 4|}{x - 4}$

(b) $f(x) = \frac{\sin(x + 4)}{x + 4}$

(c) $f(x) = \begin{cases} 1, & x \geq 4 \\ 0, & -4 < x < 4 \\ 1, & x = -4 \\ 0, & x < -4 \end{cases}$

$x = 4$ is nonremovable, $x = -4$ is removable



99. True

1. $f(c) = L$ is defined.

2. $\lim_{x \to c} f(x) = L$ exists.

3. $f(c) = \lim_{x \to c} f(x)$

All of the conditions for continuity are met.

100. True. If $f(x) = g(x), x \neq c$, then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$ (if they exist) and at least one of these limits then does not equal the corresponding function value at $x = c$.