**51.** $$\lim_{x\to4}\frac{x-4}{x^2-16}=\lim_{x\to4}\frac{x-4}{(x+4)(x-4)}=\lim_{x\to4}\frac{1}{x+4}=\frac{1}{8}$$

**52.** $$\lim_{x\to3}\frac{3-x}{x^2-9}=\lim_{x\to3}\frac{3-x}{(x-3)(x+3)}=\lim_{x\to3}\frac{-1}{x+3}=-\frac{1}{6}$$

**53.** $$\lim_{x\to4}\frac{x^2-5x+4}{x^2-2x-8}=\lim_{x\to4}\frac{(x-4)(x-1)}{(x-4)(x+2)}=\lim_{x\to4}\frac{(x-1)}{(x+2)}=\frac{3}{6}=\frac{1}{2}$$

**54.** $$\lim_{x\to-3}\frac{x^2+x-6}{x^2-9}=\lim_{x\to-3}\frac{(x+3)(x-2)}{(x+3)(x-3)}=\lim_{x\to-3}\frac{x-2}{x-3}=\frac{-5}{-6}=\frac{5}{6}$$

**55.** $$\lim_{x\to3}\frac{\sqrt{x+1}-2}{x-3}=\lim_{x\to3}\frac{\sqrt{x+1}-2}{x-3}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}=\lim_{x\to3}\frac{x-3}{(x-3)\left[\sqrt{x+1}+2\right]}=\lim_{x\to3}\frac{1}{\sqrt{x+1}+2}=\frac{1}{4}$$

**56.** $$\lim_{x\to4}\frac{\sqrt{x+5}-3}{x-4}=\lim_{x\to4}\frac{\sqrt{x+5}-3}{x-4}\cdot\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}=\lim_{x\to4}\frac{(x+5)-9}{(x-4)\left(\sqrt{x+5}+3\right)}=\lim_{x\to4}\frac{1}{\sqrt{x+5}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}$$

**57.** $$\lim_{x\to0}\frac{\sqrt{2+x}-\sqrt{2}}{x}=\lim_{x\to0}\frac{\sqrt{2+x}-\sqrt{2}}{x}\cdot\frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}}=\lim_{x\to0}\frac{2+x-2}{\left(\sqrt{2+x}+\sqrt{2}\right)x}=\lim_{x\to0}\frac{1}{\sqrt{2+x}+\sqrt{2}}=\frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}$$

**58.** $$\lim_{x\to0}\frac{\sqrt{x+5}-\sqrt{5}}{x}=\lim_{x\to0}\frac{\sqrt{x+5}-\sqrt{5}}{x}\cdot\frac{\sqrt{x+5}+\sqrt{5}}{\sqrt{x+5}+\sqrt{5}}=\lim_{x\to0}\frac{(x+5)-5}{x\left(\sqrt{x+5}+\sqrt{5}\right)}=\lim_{x\to0}\frac{1}{\sqrt{x+5}+\sqrt{5}}=\frac{1}{2\sqrt{5}}=\frac{\sqrt{5}}{10}$$

**59.** $$\lim_{x\to 0}\frac{\frac{1}{x+4}-\frac{1}{4}}{x}=\lim_{x\to 0}\frac{\frac{4-(x+4)}{4(x+4)}}{x}$$
$$=\lim_{x\to 0}\frac{-1}{4(x+4)}=-\frac{1}{16}$$

**60.** $$\lim_{x\to 0}\frac{\frac{1}{3+x}-\frac{1}{3}}{x}=\lim_{x\to 0}\frac{3-(3+x)}{(3+x)3(x)}=\lim_{x\to 0}\frac{-x}{(3+x)(3)(x)}=\lim_{x\to 0}\frac{-1}{(3+x)3}=-\frac{1}{9}$$

**61.** $$\lim_{\Delta x\to 0}\frac{2(x+\Delta x)-2x}{\Delta x}=\lim_{\Delta x\to 0}\frac{2x+2\Delta x-2x}{\Delta x}$$
$$=\lim_{\Delta x\to 0}2=2$$

**62.** $$\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}=\lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-x^2}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x(2x+\Delta x)}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x)=2x$$

**63.** $$\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x}=\lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-2x-2\Delta x+1-x^2+2x-1}{\Delta x}$$
$$=\lim_{\Delta x\to 0}(2x+\Delta x-2)=2x-2$$

**64.** $$\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x}=\lim_{\Delta x\to 0}\frac{x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3-x^3}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{\Delta x\left(3x^2+3x\Delta x+(\Delta x)^2\right)}{\Delta x}=\lim_{\Delta x\to 0}\left(3x^2+3x\Delta x+(\Delta x)^2\right)=3x^2$$

**65.** $$\lim_{x\to 0}\frac{\sin x}{5x}=\lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)\left(\frac{1}{5}\right)\right]=(1)\left(\frac{1}{5}\right)=\frac{1}{5}$$

**66.** $$\lim_{x\to 0}\frac{3(1-\cos x)}{x}=\lim_{x\to 0}\left[3\left(\frac{(1-\cos x)}{x}\right)\right]=(3)(0)=0$$

**67.** $$\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^2}=\lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1-\cos x}{x}\right]$$
$$=(1)(0)=0$$

**68.** $$\lim_{\theta\to 0}\frac{\cos\theta\tan\theta}{\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

**69.** $$\lim_{x\to 0}\frac{\sin^2 x}{x}=\lim_{x\to 0}\left[\frac{\sin x}{x}\sin x\right]=(1)\sin 0=0$$

**70.** $\lim_{x\to 0}\frac{\tan^2 x}{x} = \lim_{x\to 0}\frac{\sin^2 x}{x\cos^2 x} = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{\sin x}{\cos^2 x}\right]$

$= (1)(0) = 0$

**71.** $\lim_{\phi\to\pi}\phi\sec\phi = \pi(-1) = -\pi$

**72.** $\lim_{h\to 0}\frac{(1-\cos h)^2}{h} = \lim_{h\to 0}\left[\frac{1-\cos h}{h}(1-\cos h)\right]$

$= (0)(0) = 0$

**73.** $\lim_{x\to\pi/2}\frac{\cos x}{\cot x} = \lim_{x\to\pi/2}\sin x = 1$

**74.** $\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x} = \lim_{x\to\pi/4}\frac{\cos x-\sin x}{\sin x\cos x-\cos^2 x}$

$= \lim_{x\to\pi/4}\frac{-(\sin x-\cos x)}{\cos x(\sin x-\cos x)}$

$= \lim_{x\to\pi/4}\frac{-1}{\cos x}$

$= \lim_{x\to\pi/4}(-\sec x)$

$= -\sqrt{2}$

**75.** $\lim_{t\to 0}\frac{\sin 3t}{2t} = \lim_{t\to 0}\left(\frac{\sin 3t}{3t}\right)\left(\frac{3}{2}\right) = (1)\left(\frac{3}{2}\right) = \frac{3}{2}$

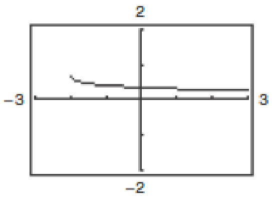
**76.** $\lim_{x\to 0}\frac{\sin 2x}{\sin 3x} = \lim_{x\to 0}\left[2\left(\frac{\sin 2x}{2x}\right)\left(\frac{1}{3}\right)\left(\frac{3x}{\sin 3x}\right)\right]$

$= 2(1)\left(\frac{1}{3}\right)(1) = \frac{2}{3}$

**77.** $f(x) = \dfrac{\sqrt{x+2} - \sqrt{2}}{x}$

| $x$ | –0.1 | –0.01 | –0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.358 | 0.354 | 0.354 | ? | 0.354 | 0.353 | 0.349 |

It appears that the limit is 0.354.




The graph has a hole at $x = 0$.

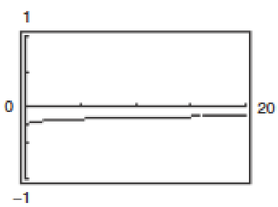
Analytically, $\lim\limits_{x\to 0} \dfrac{\sqrt{x+2} - \sqrt{2}}{x} = \lim\limits_{x\to 0} \dfrac{\sqrt{x+2} - \sqrt{2}}{x} \cdot \dfrac{\sqrt{x+2} + \sqrt{2}}{\sqrt{x+2} + \sqrt{2}}$

$$= \lim_{x\to 0} \frac{x + 2 - 2}{x\left(\sqrt{x+2} + \sqrt{2}\right)} = \lim_{x\to 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354.$$

**78.** $f(x) = \dfrac{4 - \sqrt{x}}{x - 16}$

| $x$ | 15.9 | 15.99 | 15.999 | 16 | 16.001 | 16.01 | 16.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | –0.1252 | –0.125 | –0.125 | ? | –0.125 | –0.125 | –0.1248 |

It appears that the limit is –0.125.



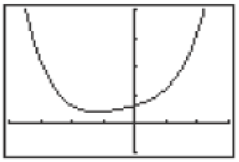

The graph has a hole at $x = 16$.

Analytically, $\lim\limits_{x\to 16} \dfrac{4 - \sqrt{x}}{x - 16} = \lim\limits_{x\to 16} \dfrac{\left(4 - \sqrt{x}\right)}{\left(\sqrt{x} + 4\right)\left(\sqrt{x} - 4\right)} = \lim\limits_{x\to 16} \dfrac{-1}{\sqrt{x} + 4} = -\dfrac{1}{8}.$

**79.** $f(x) = \dfrac{x^5 - 32}{x - 2}$

| $x$ | 1.9 | 1.99 | 1.999 | 1.9999 | 2.0 | 2.0001 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 72.39 | 79.20 | 79.92 | 79.99 | ? | 80.01 | 80.08 | 80.80 | 88.41 |

It appears that the limit is 80.




The graph has a hole at $x = 2$.

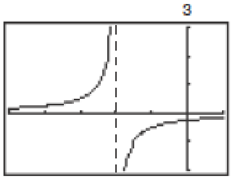
Analytically, $\lim_{x\to 2}\dfrac{x^5 - 32}{x - 2} = \lim_{x\to 2}\dfrac{(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2} = \lim_{x\to 2}(x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.$

(*Hint:* Use long division to factor $x^5 - 32$.)

**80.** $f(x) = \dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x}$

| $x$ | –0.1 | –0.01 | –0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | –0.263 | –0.251 | –0.250 | ? | –0.250 | –0.249 | –0.238 |

It appears that the limit is –0.250.



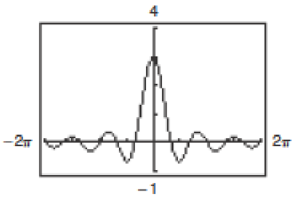

The graph has a hole at $x = 0$.

Analytically, $\lim_{x\to 0}\dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x} = \lim_{x\to 0}\dfrac{2 - (2 + x)}{2(2 + x)} \cdot \dfrac{1}{x} = \lim_{x\to 0}\dfrac{-x}{2(2 + x)} \cdot \dfrac{1}{x} = \lim_{x\to 0}\dfrac{-1}{2(2 + x)} = -\dfrac{1}{4}.$

**81.** $f(t) = \dfrac{\sin 3t}{t}$

| $t$ | $-0.1$ | $-0.01$ | $-0.001$ | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(t)$ | 2.96 | 2.9996 | 3 | ? | 3 | 2.9996 | 2.96 |

It appears that the limit is 3.



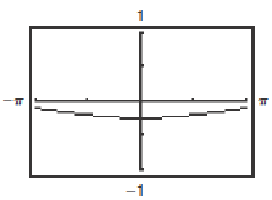

The graph has a hole at $t = 0$.

Analytically, $\lim_{t \to 0} \dfrac{\sin 3t}{t} = \lim_{t \to 0} 3\left(\dfrac{\sin 3t}{3t}\right) = 3(1) = 3.$

**82.** $f(x) = \dfrac{\cos x - 1}{2x^2}$

| $x$ | $-1$ | $-0.1$ | $-0.01$ | 0.01 | 0.1 | 1 |
|---|---|---|---|---|---|---|
| $f(x)$ | $-0.2298$ | $-0.2498$ | $-0.25$ | $-0.25$ | $-0.2498$ | $-0.2298$ |

It appears that the limit is $-0.25$.




The graph has a hole at $x = 0$.

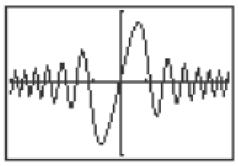
Analytically, 
$$\frac{\cos x - 1}{2x^2} \cdot \frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{2x^2(\cos x + 1)}$$
$$= \frac{-\sin^2 x}{2x^2(\cos x + 1)}$$
$$= \frac{\sin^2 x}{x^2} \cdot \frac{-1}{2(\cos x + 1)}$$

$$\lim_{x \to 0}\left[\frac{\sin^2 x}{x^2} \cdot \frac{-1}{2(\cos x + 1)}\right] = 1\left(\frac{-1}{4}\right) = -\frac{1}{4} = -0.25$$

**83.** $f(x) = \dfrac{\sin x^2}{x}$

| $x$ | –0.1 | –0.01 | –0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | –0.099998 | –0.01 | –0.001 | ? | 0.001 | 0.01 | 0.099998 |

It appears that the limit is 0.



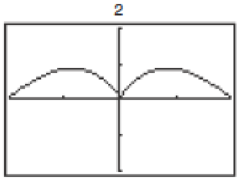

The graph has a hole at $x = 0$.

Analytically, $\lim_{x\to 0} \dfrac{\sin x^2}{x} = \lim_{x\to 0} x\left(\dfrac{\sin x^2}{x}\right) = 0(1) = 0.$

**84.** $f(x) = \dfrac{\sin x}{\sqrt[3]{x}}$

| $x$ | –0.1 | –0.01 | –0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.215 | 0.0464 | 0.01 | ? | 0.01 | 0.0464 | 0.215 |

It appears that the limit is 0.




The graph has a hole at $x = 0$.

Analytically, $\lim_{x\to 0} \dfrac{\sin x}{\sqrt[3]{x}} = \lim_{x\to 0} \sqrt[3]{x^2}\left(\dfrac{\sin x}{x}\right) = (0)(1) = 0.$

**85.** $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{3(x+\Delta x) - 2 - (3x-2)}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{3x + 30x - 2 - 3x + 2}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{3\Delta x}{\Delta x} = 3$

**86.** $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \dfrac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$

$$= \lim_{\Delta x\to 0} \frac{x + \Delta x - x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)} = \lim_{\Delta x\to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

**87.** $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{x + \Delta x + 3} - \frac{1}{x + 3}}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{x + 3 - (x + \Delta x + 3)}{(x + \Delta x + 3)(x + 3)} \cdot \frac{1}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{-\Delta x}{(x + \Delta x + 3)(x + 3)\Delta x}$

$= \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + 3)(x + 3)} = \frac{-1}{(x + 3)^2}$

**88.** $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 4(x + \Delta x) - (x^2 - 4x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - 4x - 4\Delta x - x^2 + 4x}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x - 4)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 4) = 2x - 4$

**89.** $\lim_{x \to 0} (4 - x^2) \le \lim_{x \to 0} f(x) \le \lim_{x \to 0} (4 + x^2)$

$4 \le \lim_{x \to 0} f(x) \le 4$

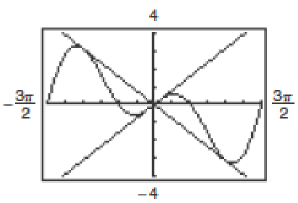
Therefore, $\lim_{x \to 0} f(x) = 4$.

**90.** $\lim_{x \to a} [b - |x - a|] \le \lim_{x \to a} f(x) \le \lim_{x \to a} [b + |x - a|]$

$b \le \lim_{x \to a} f(x) \le b$

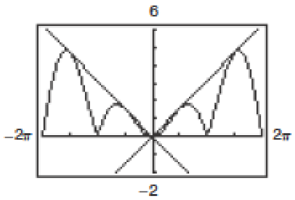
Therefore, $\lim_{x \to a} f(x) = b$.

**91.** $f(x) = x \cos x$



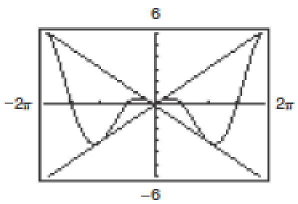

$\lim_{x \to 0} (x \cos x) = 0$

**92.** $f(x) = |x \sin x|$



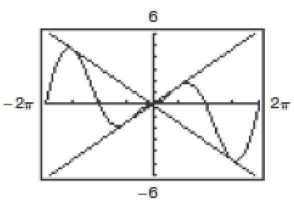

$\lim_{x \to 0} |x \sin x| = 0$

**93.** $f(x) = |x| \cos x$



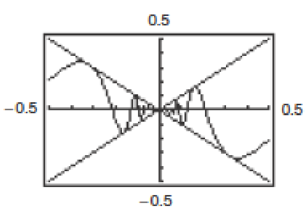

$\lim_{x \to 0} |x| \cos x = 0$

**94.** $f(x) = |x| \sin x$



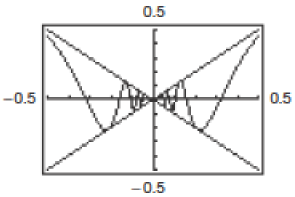

$\lim_{x \to 0} |x| \sin x = 0$

**95.** $h(x) = x \cos \frac{1}{x}$




$\lim_{x \to 0} \left( x \cos \frac{1}{x} \right) = 0$

**96.** $f(x) = x \sin \frac{1}{x}$




$\lim_{x \to 0}\left(x \sin \frac{1}{x}\right) = 0$

**97.** You say that two functions *f* and *g* agree at all but one point (on an open interval) if $f(x) = g(x)$ for all *x* in the interval except for $x = c$, where *c* is in the interval.

**98.** $f(x) = \frac{x^2 - 1}{x - 1}$ and $g(x) = x + 1$ agree at all points except $x = 1$.

**99.** An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

for which $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$

**100.** If a function *f* is squeezed between two functions *h* and *g*, $h(x) \le f(x) \le g(x)$, and *h* and *g* have the same limit *L* as $x \to c$, then $\lim_{x \to c} f(x)$ exists and equals *L*.