51. $\lim_{x\to 6}|x-6| = |6-6| = 0$

Given $\varepsilon > 0$: $\big||x-6|-0\big| < \varepsilon$

$|x-6| < \varepsilon$

So, let $\delta = \varepsilon$.

So for $|x-6| < \delta = \varepsilon$, you have

$|x-6| < \varepsilon$

$\big||x-6|-0\big| < \varepsilon$

$|f(x)-L| < \varepsilon.$

52. $\lim_{x\to -5}|x-5| = |(-5)-5| = |-10| = 10$

Given $\varepsilon > 0$: $\big||x-5|-10\big| < \varepsilon$

$|-(x-5)-10| < \varepsilon \quad (x-5 < 0)$

$|-x-5| < \varepsilon$

$|x-(-5)| < \varepsilon$

So, let $\delta = \varepsilon$.

So for $|x-(-5)| < \delta = \varepsilon$, you have

$|-(x+5)| < \varepsilon$

$|-(x-5)-10| < \varepsilon$

$\big||x-5|-10\big| < \varepsilon \quad (\text{because } x-5 < 0)$

$|f(x)-L| < \varepsilon.$

53. $\lim_{x \to -3} (x^2 + 3x) = 0$

Given $\varepsilon > 0$:

$$\left|(x^2 + 3x) - 0\right| < \varepsilon$$
$$\left|x(x + 3)\right| < \varepsilon$$
$$\left|x + 3\right| < \frac{\varepsilon}{|x|}$$

If you assume $-4 < x < -2$, then $\delta = \varepsilon/4$.

So for $0 < \left|x - (-3)\right| < \delta = \frac{\varepsilon}{4}$, you have

$$\left|x + 3\right| < \frac{1}{4}\varepsilon < \frac{1}{|x|}\varepsilon$$
$$\left|x(x + 3)\right| < \varepsilon$$
$$\left|x^2 + 3x - 0\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

54. $\lim_{x \to 1} (x^2 + 1) = 2$

Given $\varepsilon > 0$:

$$\left|(x^2 + 1) - 2\right| < \varepsilon$$
$$\left|x^2 - 1\right| < \varepsilon$$
$$\left|(x + 1)(x - 1)\right| < \varepsilon$$
$$\left|x - 1\right| < \frac{\varepsilon}{|x + 1|}$$

If you assume $0 < x < 2$, then $\delta = \varepsilon/3$.

So for $0 < \left|x - 1\right| < \delta = \frac{\varepsilon}{3}$, you have

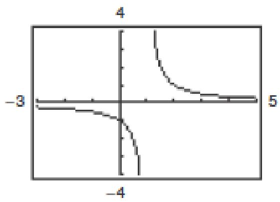
$$\left|x - 1\right| < \frac{1}{3}\varepsilon < \frac{1}{|x + 1|}\varepsilon$$
$$\left|x^2 - 1\right| < \varepsilon$$
$$\left|(x^2 + 1) - 2\right| < \varepsilon$$
$$\left|f(x) - 2\right| < \varepsilon.$$

55. $\lim_{x \to \pi} f(x) = \lim_{x \to \pi} 4 = 4$

56. $\lim_{x \to \pi} f(x) = \lim_{x \to \pi} x = \pi$

57. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

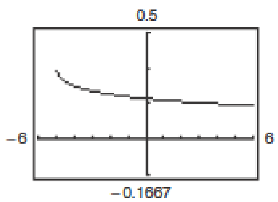
$\lim_{x \to 3} f(x) = \dfrac{1}{2}$



The domain is all $x \neq 1, 3$. The graphing utility does not show the hole at $\left(3, \frac{1}{2}\right)$.

58. $f(x) = \dfrac{\sqrt{x + 5} - 3}{x - 4}$

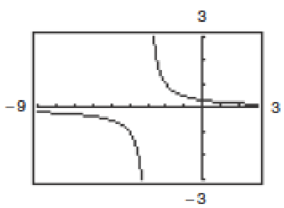
$\lim_{x \to 4} f(x) = \dfrac{1}{6}$



The domain is $[-5, 4) \cup (4, \infty)$. The graphing utility does not show the hole at $\left(4, \frac{1}{6}\right)$.

59. $f(x) = \dfrac{x - 3}{x^2 - 9}$

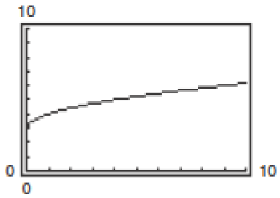
$\lim_{x \to 3} f(x) = \dfrac{1}{6}$



The domain is all $x \neq \pm 3$. The graphing utility does not show the hole at $\left(3, \frac{1}{6}\right)$.

60. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

$\lim_{x \to 9} f(x) = 6$

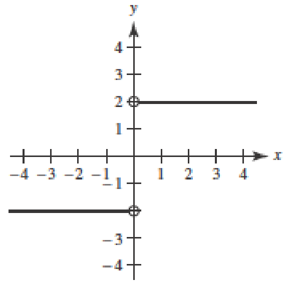


The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

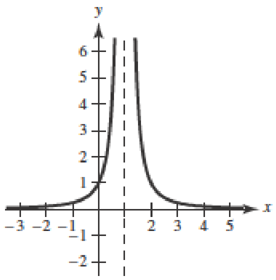
61. $\lim_{x \to 8} f(x) = 25$ means that the values of f approach 25 as x gets closer and closer to 8.

62. In the definition of $\lim_{x \to c} f(x)$, f must be defined on both sides of c, but does not have to be defined at c itself. The value of f at c has no bearing on the limit as x approaches c.

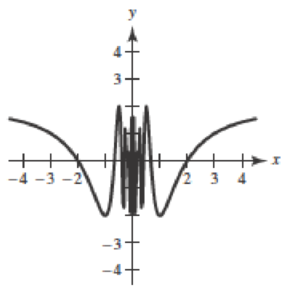
63. (i) The values of f approach different numbers as x approaches c from different sides of c:



(ii) The values of f increase without bound as x approaches c:



(iii) The values of f oscillate between two fixed numbers as x approaches c:



64. (a) No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as x approaches 2.

(b) No. The fact that $\lim_{x \to 2} f(x) = 4$ has no bearing on the value of f at 2.

65. (a) $C = 2\pi r$

$$r = \frac{C}{2\pi} = \frac{6}{2\pi} = \frac{3}{\pi} \approx 0.9549 \text{ cm}$$

(b) When $C = 5.5: r = \frac{5.5}{2\pi} \approx 0.87535$ cm

When $C = 6.5: r = \frac{6.5}{2\pi} \approx 1.03451$ cm

So $0.87535 < r < 1.03451$.

(c) $\lim_{x \to 3/\pi} (2\pi r) = 6; \varepsilon = 0.5; \delta \approx 0.0796$

66. $V = \frac{4}{3}\pi r^3, V = 2.48$

(a) $2.48 = \frac{4}{3}\pi r^3$

$$r^3 = \frac{1.86}{\pi}$$

$$r \approx 0.8397 \text{ in.}$$

(b)

$$2.45 \le V \le 2.51$$

$$2.45 \le \frac{4}{3}\pi r^3 \le 2.51$$

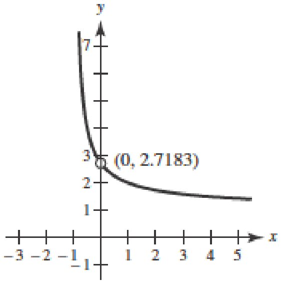
$$0.5849 \le r^3 \le 0.5992$$

$$0.8363 \le r \le 0.8431$$

(c) For $\varepsilon = 2.51 - 2.48 = 0.03, \delta \approx 0.003$

67. $f(x) = (1 + x)^{1/x}$

$\lim_{x \to 0}(1 + x)^{1/x} = e \approx 2.71828$



x	$f(x)$
–0.1	2.867972
–0.01	2.731999
–0.001	2.719642
–0.0001	2.718418
–0.00001	2.718295
–0.000001	2.718283

x	$f(x)$
0.1	2.593742
0.01	2.704814
0.001	2.716942
0.0001	2.718146
0.00001	2.718268
0.000001	2.718280

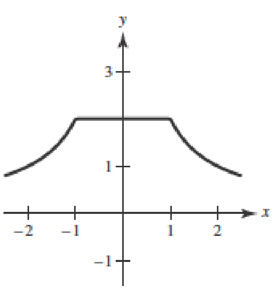
68. $f(x) = \dfrac{|x + 1| - |x - 1|}{x}$

x	–1	–0.5	–0.1	0	0.1	0.5	1.0
$f(x)$	2	2	2	Undef.	2	2	2

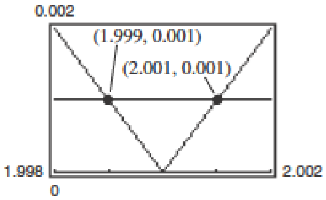
$\lim_{x \to 0} f(x) = 2$

Note that for

$$-1 < x < 1, x \neq 0, f(x) = \frac{(x + 1) + (x - 1)}{x} = 2.$$



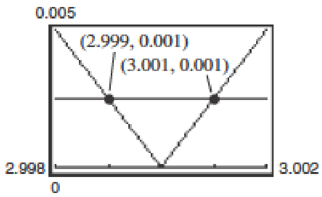
69.



Using the zoom and trace feature, $\delta = 0.001$. So $(2 - \delta, 2 + \delta) = (1.999, 2.001)$.

Note: $\dfrac{x^2 - 4}{x - 2} = x + 2$ for $x \neq 2$.

70.



From the graph, $\delta = 0.001$. So $(3 - \delta, 3 + \delta) = (2.999, 3.001)$.

Note: $\dfrac{x^2 - 3x}{x - 3} = x$ for $x \neq 3$.

71. False. The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as $x \to c$.

72. True

73. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$$f(2) = 0$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2}(x - 4) = 2 \neq 0$$

74. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2}(x - 4) = 2 \text{ and } f(2) = 0 \neq 2$$

75. $f(x) = \sqrt{x}$

$\lim_{x\to 0.25} \sqrt{x} = 0.5$ is true.

As x approaches $0.25 = \frac{1}{4}$ from either side,

$f(x) = \sqrt{x}$ approaches $\frac{1}{2} = 0.5$.

76. $f(x) = \sqrt{x}$

$\lim_{x\to 0} \sqrt{x} = 0$ is false.

$f(x) = \sqrt{x}$ is not defined on an open interval containing 0 because the domain of f is $x \geq 0$.

77. Using a graphing utility, you see that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{\sin 2x}{x} = 2, \text{ etc.}$$

So, $\lim_{x\to 0} \frac{\sin nx}{x} = n$.

78. Using a graphing utility, you see that

$$\lim_{x\to 0} \frac{\tan x}{x} = 1$$

$$\lim_{x\to 0} \frac{\tan 2x}{x} = 2, \quad \text{etc.}$$

So, $\lim_{x\to 0} \frac{\tan(nx)}{x} = n$.

79. If $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, then for every $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon$. Let δ equal the smaller of δ_1 and δ_2. Then for $|x - c| < \delta$, you have $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon$. Therefore, $|L_1 - L_2| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

80. $f(x) = mx + b$, $m \neq 0$. Let $\varepsilon > 0$ be given. Take

$$\delta = \frac{\varepsilon}{|m|}.$$

If $0 < |x - c| < \delta = \dfrac{\varepsilon}{|m|}$, then

$$|m||x - c| < \varepsilon$$
$$|mx - mc| < \varepsilon$$
$$|(mx + b) - (mc + b)| < \varepsilon$$

which shows that $\lim_{x \to c} (mx + b) = mc + b$.

81. $\lim_{x \to c} [f(x) - L] = 0$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$0 < |x - c| < \delta,$$

then

$$|(f(x) - L) - 0| < \varepsilon.$$

This means the same as $|f(x) - L| < \varepsilon$ when

$$0 < |x - c| < \delta.$$

So, $\lim_{x \to c} f(x) = L$.

82. (a) $(3x+1)(3x-1)x^2 + 0.01 = (9x^2-1)x^2 + \frac{1}{100}$

$$= 9x^4 - x^2 + \frac{1}{100}$$

$$= \frac{1}{100}(10x^2-1)(90x^2-1)$$

So, $(3x+1)(3x-1)x^2 + 0.01 > 0$ if

$10x^2 - 1 < 0$ and $90x^2 - 1 < 0$.

Let $(a, b) = \left(-\frac{1}{\sqrt{90}}, \frac{1}{\sqrt{90}}\right)$.

For all $x \neq 0$ in (a, b), the graph is positive. You can verify this with a graphing utility.p

(b) You are given $\lim_{x \to c} g(x) = L > 0$. Let

$\varepsilon = \frac{1}{2}L$. There exists $\delta > 0$ such that

$0 < |x - c| < \delta$ implies that

$|g(x) - L| < \varepsilon = \frac{L}{2}$. That is,

$$-\frac{L}{2} < g(x) - L < \frac{L}{2}$$

$$\frac{L}{2} < \quad g(x) \quad < \frac{3L}{2}$$

For x in the interval $(c - \delta, c + \delta)$, $x \neq c$, you have $g(x) > \frac{L}{2} > 0$, as desired.

83. Answers will vary.

84. $\lim_{x \to 4} \frac{x^2 - x - 12}{x - 4} = 7$

n	$4 + [0.1]^n$	$f(4 + [0.1]^n)$
1	4.1	7.1
2	4.01	7.01
3	4.001	7.001
4	4.0001	7.0001

n	$4 - [0.1]^n$	$f(4 - [0.1]^n)$
1	3.9	6.9
2	3.99	6.99
3	3.999	6.999
4	3.9999	6.9999

85. The radius OP has a length equal to the altitude z of the triangle plus $\frac{h}{2}$. So, $z = 1 - \frac{h}{2}$.

$$\text{Area triangle} = \frac{1}{2}b\left(1 - \frac{h}{2}\right)$$

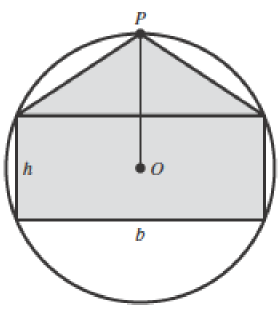
$$\text{Area rectangle} = bh$$

Because these are equal, $\frac{1}{2}b\left(1 - \frac{h}{2}\right) = bh$

$$1 - \frac{h}{2} = 2h$$

$$\frac{5}{2}h = 1$$

$$h = \frac{2}{5}.$$

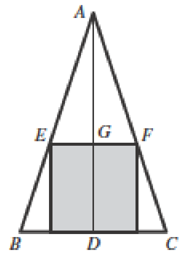


86. Consider a cross section of the cone, where EF is a diagonal of the inscribed cube. $AD = 3$, $BC = 2$. Let x be the length of a side of the cube. Then $EF = x\sqrt{2}$.

By similar triangles,

$$\frac{EF}{BC} = \frac{AG}{AD}$$

$$\frac{x\sqrt{2}}{2} = \frac{3 - x}{3}$$



Solving for x, $3\sqrt{2}x = 6 - 2x$

$$\left(3\sqrt{2} + 2\right)x = 6$$

$$x = \frac{6}{3\sqrt{2} + 2} = \frac{9\sqrt{2} - 6}{7} \approx 0.96.$$